Competitive Programming

Session 1:
Combinatorial Game Theory

Nim

- There are *n* piles of coins. Two players take turns. Each player chooses a pile, and removes any number of coins from the pile. The one who takes the last coin wins.

Let’s Play!

We’ll let you go first. 🤷🏾‍♂️

Play Nim Online

https://www.goobix.com/games/nim/

Combinatorial Games

- Turn-based multi-player games
- Can be simple win-or-lose games, or can involve points
- Everyone has *perfect information*
- Each turn, the player changes the current “state” using a valid “move”
- If a player has no valid moves, he loses
- *The game ends in a finite number of moves*

Combinatorial Games

- **No random moves**
  - Rules out all card games
- **No hidden moves**
  - Rules out games like Battleship
- **No draws in a finite number of moves**
  - Rules out games like tic-tac-toe

Combinatorial Games

- **Normal play**: The player who makes the last move wins
- **Misère play**: The player who makes the last move loses

Example

- Settings: There are n stones in a pile. Two players take turns and remove 1 or 3 stones at a time. The one who takes the last stone wins. Find out the winner if both players play perfectly

- **State space:** Each state can be represented by the number of remaining stones in the pile

- Valid moves from state $\mathscr{X}$: $\mathscr{X}$ -> ($\mathscr{X}$ – 1) or $\mathscr{X}$ -> ($\mathscr{X}$ – 3), as long as the resulting number is non-negative

- **State 0 is the losing state**

P-positions and N-positions

- Each position in a game is either a P-position or an N-position
- **P-position**: Positions that are winning for the **P**revious player (the player who just moved)
- **N-position**: Positions that are winning for the **N**ext player (the player who is about to move)

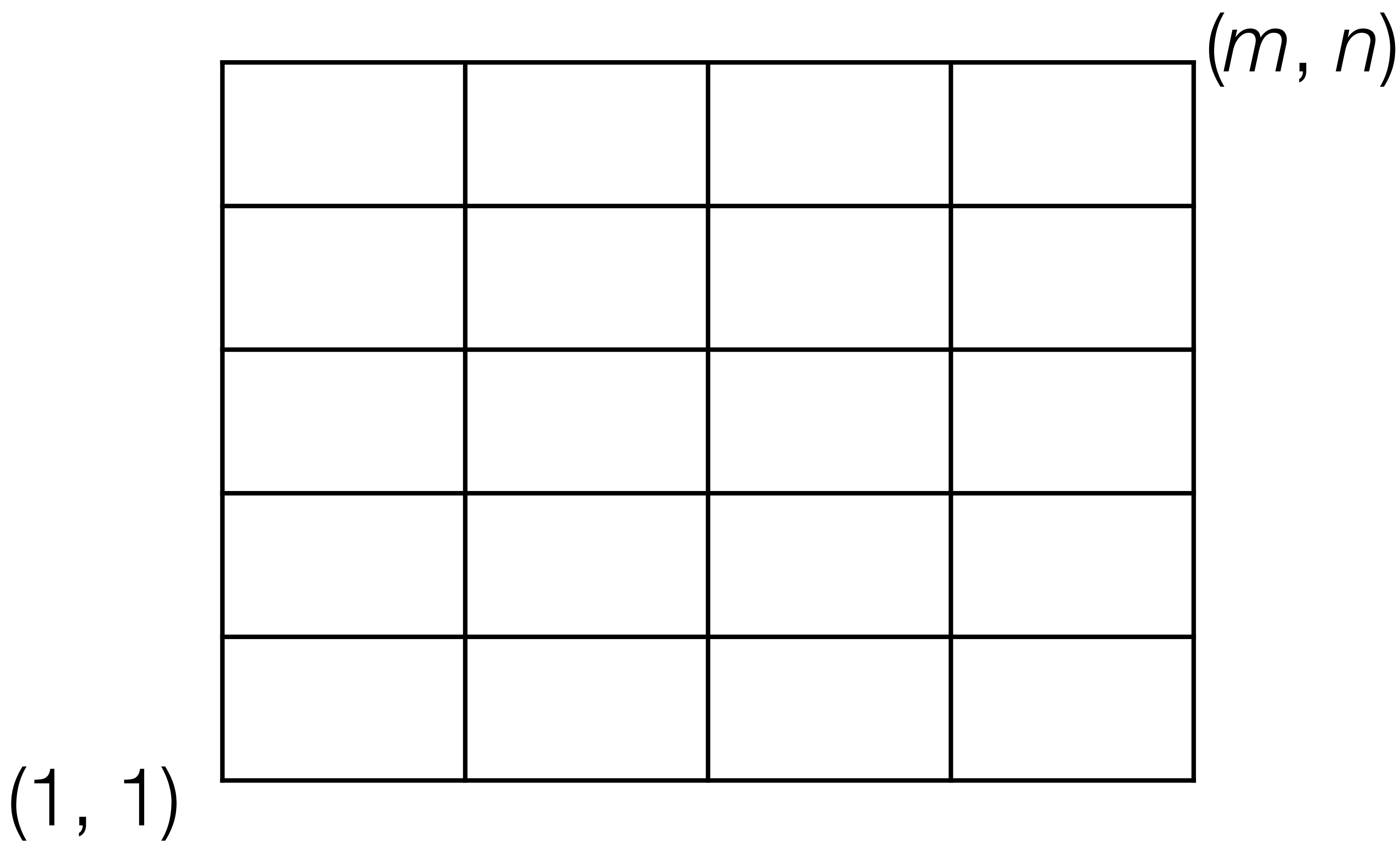
0	1	2	3	4	5	6	7	8
P	N	P	N	P	N	P	N	P

Properties

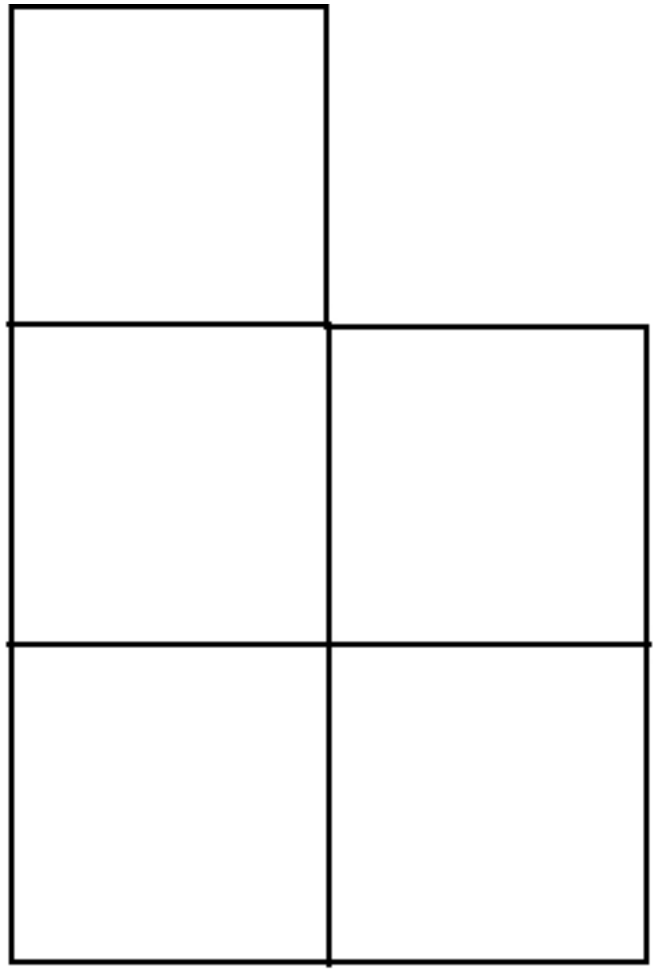
- **All terminal positions are P-positions**
- **From every N-position, there is at least one move to a P-position**
- **From every P-position, every move is to an N-position**

Chomp!

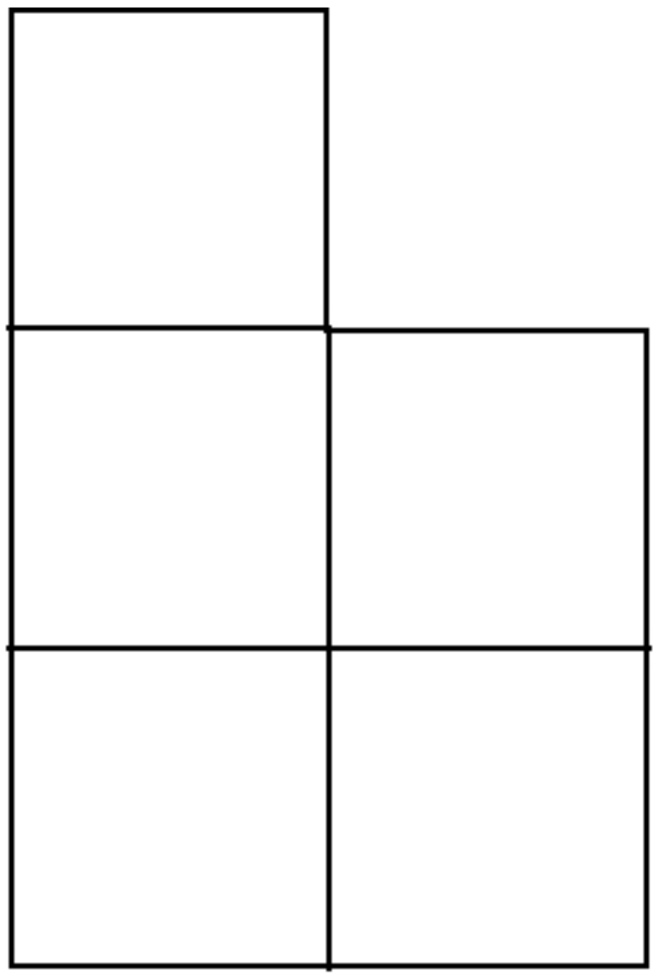
- A two-player game where each move consists of taking a square and removing it and all squares to the right and above. The player who takes (1, 1) loses.



What position is this?

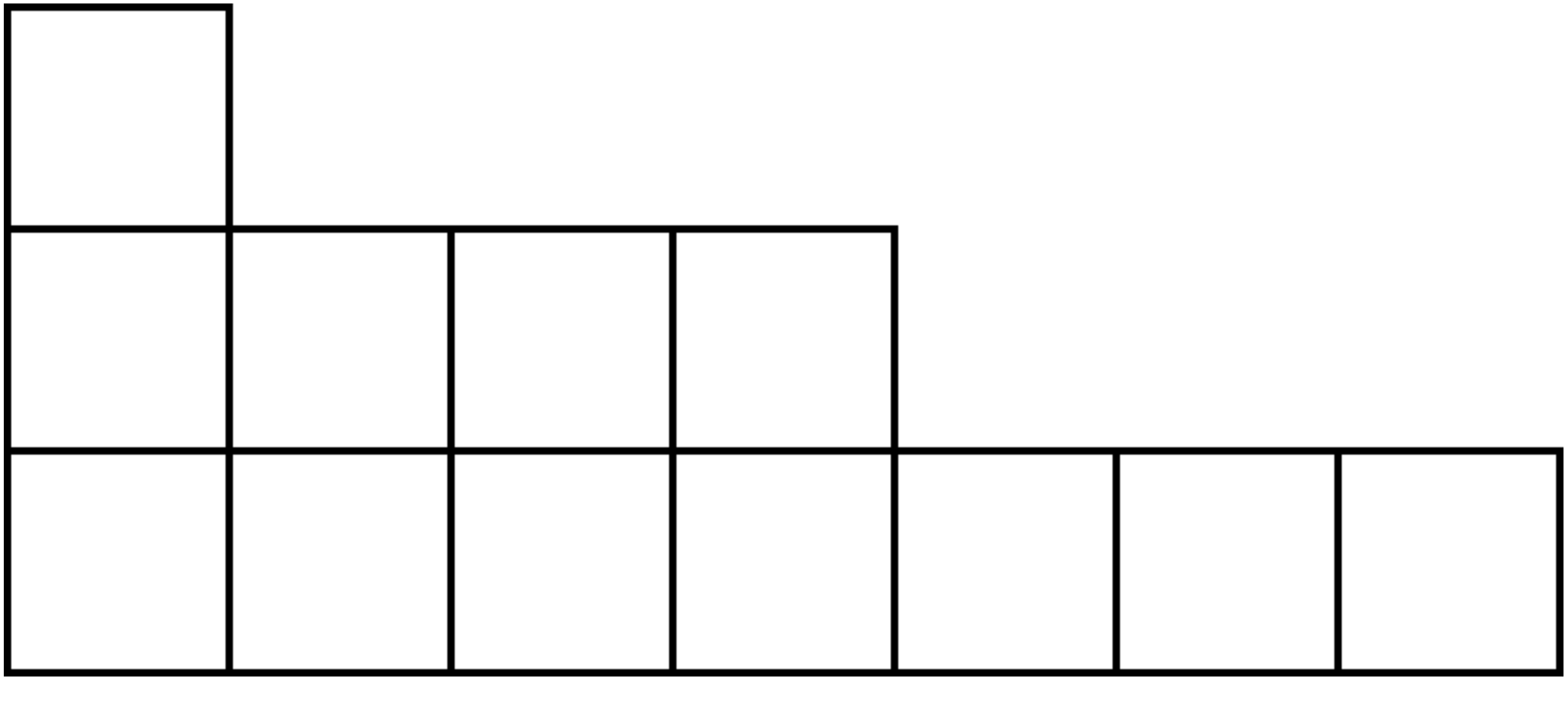


What position is this?



P Position

What position is this?



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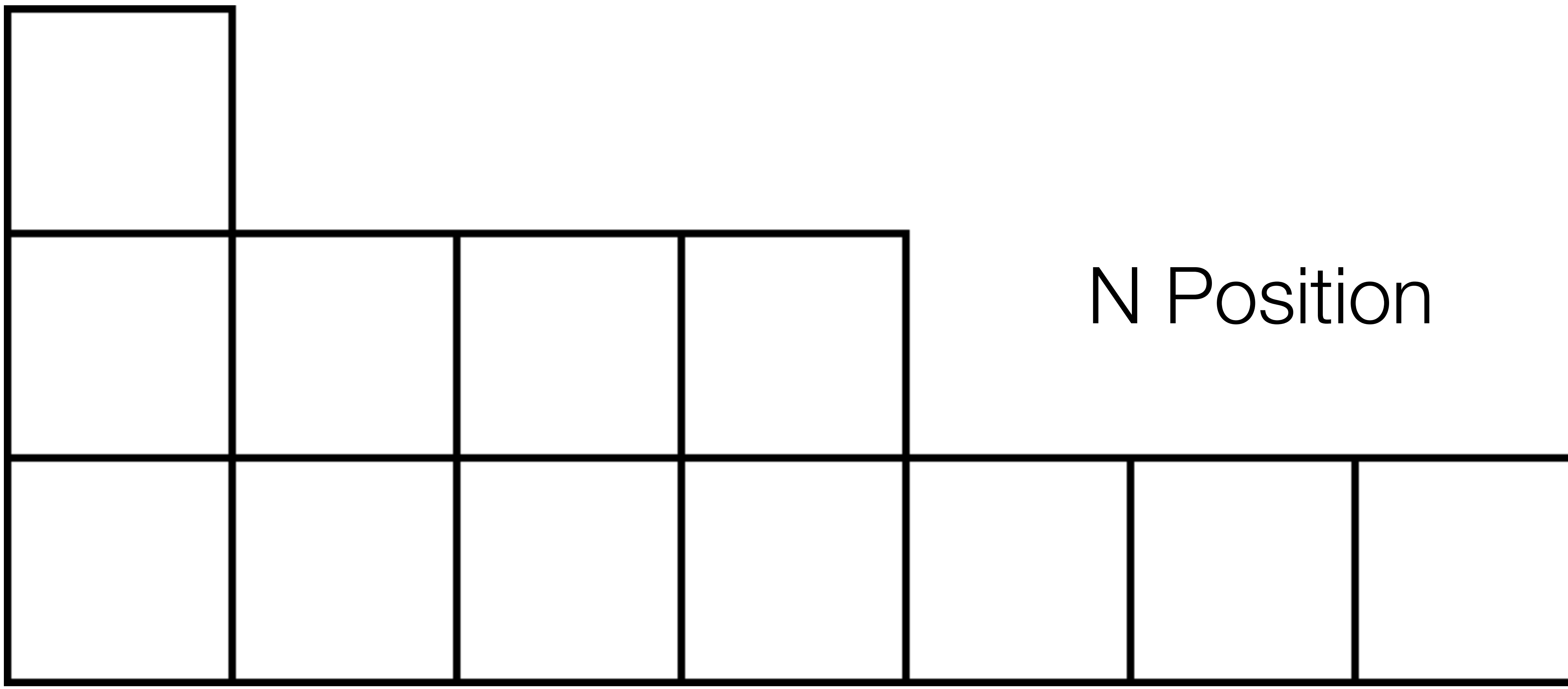

Grundy Numbers (Nimbers)

- Suppose that multiple games are played at the same time. At each turn, the player chooses a game and make a move. You lose if there is no possible move. We want to determine the winner.
- For each game, we compute its *Grundy number*
- The first player wins if and only if the XOR of all the Grundy numbers is nonzero

Computing Grundy Numbers

- Let S be a state, and T_1, T_2, …, T_m be states that can be reached from S using a single move
- The Grundy number $g(S)$ of S is the smallest non-negative integer that doesn't appear in $\{g(T_1), g(T_2), \ldots, g(T_m)\}$
- The Grundy number of a losing state is 0

Revisiting Nim

Nim Game

- Settings: There are *n* piles of coins. Two players take turns. Each player chooses a pile, and removes any number of coins from the pile. The one who takes the last coin wins. Find out the winner if both players play perfectly.

Solution to Nim

Given piles of size $a_1, a_2, a_3, \ldots, a_n$, the first player wins only if the *nim-sum* $a_1 \oplus a_2 \oplus a_3 \oplus \ldots \oplus a_n$ is non-zero.

Why?

- If the nim-sum is zero, then whatever the current player does, the nim-sum of the next state is non-zero
- If the nim-sum is non-zero, it is possible to force it to become zero

Game of Stones

www.hackerrank.com/challenges/game-of-stones-1

Tower Breakers, Revisited

www.hackerrank.com/challenges/tower-breakers-revisited-1

Zero-Move Nim

www.hackerrank.com/challenges/zero-move-nim

End of session

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